Test #2 Solutions

Q1 $\quad y = \sum_{k=0}^{\infty} a_k x^k$

$$(25-x^2)y'' + 2y = (25-x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + 2\sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=2}^{\infty} 25k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k x^k + \sum_{k=0}^{\infty} 2a_k x^k$$

$$= \sum_{k=0}^{\infty}\left(25(k+2)(k+1)a_{k+2} - k(k-1)a_k + 2a_k\right)x^k = 0$$

$$25(k+1)(k+2)a_{k+2} = (k^2-k-2)a_k = (k-2)(k+1)a_k$$

$$a_{k+2} = \frac{(k-2)}{25(k+2)}a_k \quad ②$$

$a_0 = 1 \qquad a_2 = \frac{1}{25}\frac{(-2)}{2} = \frac{-1}{25} \qquad a_4 = a_6 = \cdots = 0 \qquad$ and y_1 ②

$a_1 = 1 \qquad a_3 = \frac{(-1)}{25\cdot 3} \qquad a_5 = \frac{(1)}{25\cdot 5}a_3 = \frac{(1)(-1)}{3\cdot 5\cdot 25^2} \qquad$ and y_2 ②

$$a_7 = \frac{(3)}{25\cdot 7}a_5 = \frac{(3)(1)(-1)}{25^3\, 3\cdot 5\cdot 7}, \quad \ldots$$

This gives two linearly independent solutions

$$y_1 = 1 - \frac{1}{25}x^2$$

$$y_2 = x - \frac{1}{3\cdot 25}x^3 - \frac{1}{3\cdot 5\cdot 25^2}(1)x^5 - \frac{(1)(3)}{3\cdot 5\cdot 7\cdot 25^3}x^7 - \frac{(1)(3)(5)}{3\cdot 5\cdot 7\cdot 9\cdot 25^4}x^9 - \cdots$$

$$= x - \frac{1}{3\cdot 25}x^3 - \frac{1}{3\cdot 5\cdot 25^2}x^5 - \frac{1}{5\cdot 7\cdot 25^3}x^7 - \frac{1}{7\cdot 9\cdot 25^4}x^9 - \frac{1}{9\cdot 11\cdot 25^5}x^{11} - \cdots$$

The general solution is $y = Ay_1 + By_2$. Then initial value problem $y(0) = 1$, $y'(0) = 1$ gives $A = B = 1$, so

$$y = 1 + x - \frac{x^2}{25} - \frac{x^3}{3\cdot 25} - \frac{x^5}{3\cdot 5\cdot 25^2} - \frac{x^7}{5\cdot 7\cdot 25^3} - \frac{x^9}{7\cdot 9\cdot 25^4} - \cdots$$

b) The term containing x^{57} is $-\frac{x^{57}}{25^{28}\, 55\cdot 57} = \frac{y^{(57)}(0)}{57!}$

(2) $y^{(57)}(0) = \frac{-57!}{55\cdot 57\cdot 25^{28}} = \frac{-56!}{55\cdot 25^{28}}$

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$$y = \sum_{k=0}^{\infty} a_k x^{r+k}$$

$$2xy'' + (x+1)y' + 3y$$

$$= 2x \sum_{k=0}^{\infty} (r+k)(r+k-1) a_k x^{r+k-2} + (x+1) \sum_{k=0}^{\infty} (r+k) a_k x^{r+k-1} + 3 \sum_{k=0}^{\infty} a_k x^{r+k}$$

$$= \sum_{k=0}^{\infty} 2(r+k)(r+k-1) a_k x^{r+k-1} + \sum_{k=0}^{\infty} (r+k) a_k x^{r+k} + \sum_{k=0}^{\infty} (r+k) a_k x^{r+k-1} + \sum_{k=0}^{\infty} 3 a_k x^{r+k}$$

$$= \sum_{k=-1}^{\infty} 2(r+k+1)(r+k) a_{k+1} x^{r+k} + \sum_{k=0}^{\infty} (r+k) a_k x^{r+k} + \sum_{k=-1}^{\infty} (r+k+1) a_{k+1} x^{r+k} + \sum_{k=0}^{\infty} 3 a_k x^{r+k}$$

$$= 2(r)(r-1) a_0 x^{r-1} + (r) a_0 x^{r-1}$$

$$+ \sum_{k=0}^{\infty} \Big(\big(2(r+k+1)(r+k) + (r+k+1) \big) a_{k+1} + (r+k+3) a_k \Big) x^{r+k}$$

$$= (2r^2 - r) a_0 x^{r-1} + \sum_{k=0}^{\infty} \Big((r+k+1)(2r+2k+1) a_{k+1} + (r+k+3) a_k \Big) x^{r+k}$$

$2r^2 - r = r(2r-1) = 0$ so $r = 0$ or $r = \frac{1}{2}$

$r = 0$: $\quad a_{k+1} = \dfrac{-(k+3)}{(k+1)(2k+1)}$ or $a_k = -\dfrac{(k+2)}{k(2k-1)}$

$r = \frac{1}{2}$: $\quad a_{k+1} = -\dfrac{(\frac{1}{2}+k+3)}{(\frac{1}{2}+k+1)(1+2k+1)} a_k = -\dfrac{(2k+7)}{(2k+3)(2k+2)} a_k$